

THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010 I/J University Mathematics 2015-2016
Assignment 2

1. Evaluate each of the following limits.

- (a) $\lim_{x \rightarrow +\infty} \frac{3x}{x-1} - \frac{2x}{x+1}$
(b) $\lim_{x \rightarrow +\infty} (\sqrt{x+1} - \sqrt{x})\sqrt{x+2}$
(c) $\lim_{x \rightarrow +\infty} \frac{\sqrt{x + \sqrt{x + \sqrt{x}}}}{\sqrt{x+1}}$
(d) $\lim_{x \rightarrow +\infty} \left(\frac{x-1}{x+2}\right)^{2x}$
(e) $\lim_{x \rightarrow 0} \frac{\tan 3x}{2x}$
(f) $\lim_{x \rightarrow 0} \frac{(1+x)^n - 1}{x}$, where n is a natural number.

2. Let $f(x) = \sqrt{e^{-\frac{1}{x}}}$ for $x \neq 0$.

- (a) Do $\lim_{x \rightarrow 0^+} f(x)$ and $\lim_{x \rightarrow 0^-} f(x)$ exist?
(b) Does $\lim_{x \rightarrow 0} f(x)$ exist?

3. Let $f(x) = \sin(\ln x)$ for $x > 0$. Show that $\lim_{x \rightarrow 0^+} f(x)$ does not exist.

(Hint: Consider $a_n = e^{-(2n-\frac{1}{2})\pi}$ and $b_n = e^{-(2n+\frac{1}{2})\pi}$.)

4. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x \cos\left(\frac{1}{e^x - e^{-x}}\right) & \text{if } x \neq 0, \\ a & \text{if } x = 0, \end{cases}$$

where a is a real number.

- (a) Find $\lim_{x \rightarrow 0} f(x)$.
(b) If $f(x)$ is continuous at $x = 0$, find the value of a .
5. Let $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function such that $0 \leq f(x) \leq 1$ for all $x \in [0, 1]$. Show that there exists $c \in [0, 1]$ such that $f(c) = c$. (Hint: Consider the function $g(x) = f(x) - x$.)
6. Let $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function that satisfies $f(xy) = f(x)f(y)$ for all $x, y \in [0, 1]$.
- (a) Show that $f(0) = 0$ or $f(x) = 1$ for all $x \in [0, 1]$.
(b) Suppose that $f(0) = 0$.

(i) Let $x \in [0, 1)$. By using the mathematical induction, show that

$$[f(x)]^{2^n} = f(x^{2^n}) \quad \text{---} (*)$$

for all natural numbers n

(ii) By taking limit on both sides of (*), show that $-1 < f(x) < 1$ for all $x \in [0, 1)$.